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Class Notes.

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1) Functions

let D & R be sets

Then $F: D \mapsto R$ denotes a function, given input $d \in D$.
domain — codomain/range.

↳ $f(x)$ argument

($e.g.$ abs: $\mathbb{Z} \mapsto \mathbb{Z}$, e.g. $abs(-2) = 2$)
↳ Q: is abs Fcn onto? $\forall r \in R, \exists d \in D$ s.t. $f(d) = r$
NOT ONTO

example

$f: \{00, 01, 10, 11\} \mapsto \{0, 1\}$

s.t. AND FUNCTION

$x \in D$	$f(x) \in R$
0 0	0
0 1	0
1 0	0
1 1	1

- k-ary function - Fcn taking k arguments
i.e. $f: A_1 \times A_2 \times \dots \times A_k \mapsto R$
- k is "arity"

a) predicate - Fcn whose codomain is $\{T, F\}$ or $\{0, 1\}$

e.g. fast: {Ferrari, bicycle} $\mapsto$ {T, F}

fast (Ferrari) = T

b) relation - a predicate whose domain is $A^k = \underbrace{A \times \dots \times A}_{k \text{ times}}$

i.e. $f: A^k \mapsto \{T, F\}$

example #1

let A = set of all ppl on Earth

married (Brad Pitt, Angelina Jolie) = 0

example #2

let $A = \mathbb{Z}$

equals $(5, 4) = 0$

↳ can also describe predicate with sets

e.g. married = {(Homer S, Marge S), ...}

equals = $\{(3,3), (4,4), \dots\}$

c) Equivalence relation - binary relation (2-tuple) $R: A^2 \mapsto \{T, F\}$ s.t.

1) Reflexive : $\forall x \in A$, $R(x, x) = T$

2) Symmetric : $\forall x, y \in A$, $R(x, y) = R(y, x)$

3) Transitive : $\forall x, y, z \in A$, $R(x, y) \wedge R(y, z) \Rightarrow R(x, z)$
AND — IMPLIES

example of equivalence relations

e.g. equals is an equivalent rel'n'!

married is not an equivalence rel'n'!

2. Strings & languages

- Alphabet - a non-empty, finite set

e.g. $\Sigma = \{a, b, c, d, \dots, z\}$ $\Sigma = \{0, 1\}$

string over alphabet - finite sequence of symbols from Σ

e.g. $\Sigma = \{0,1\}$ $s = 001101$ is a string over Σ

↳ length ⇒ $|s|$ of string is # of symbols it contains

e.g. s = squirrel, $|s| = 8$

a) special strings

↳ empty string : ε ⇒ $|\varepsilon| = 0$

ε is a substring of any string

↳ substring : s' is a substring of s if s' appears consecutively within s.

e.g. s = catinthe s' = cat

↳ concatenation : given x & y, concatenation is xy (xy ≠ yx)

e.g. x = cat & y = dog ⇒ xy = catdog

↳ prefix : x is a prefix of y if y = xz for some string z.

e.g. $y = \underbrace{\text{cat}}_{x}\underbrace{\text{inthe}}_{z}$ x = cat.

↳ language : a set of strings

e.g. L = English (or any other languages)

$L = \{0, 00, 1\}$

L = { s | s is a set of all cities in Europe with a tour costing less than (≤) €10k }

3. Boolean logic

1) Negation ($\neg$) : $\neg 0 = 1$ $\neg 1 = 0$

2) conjunction / AND ($\wedge$) :

$0 \wedge 0 = 0$
$0 \wedge 1 = 0$
$1 \wedge 0 = 0$
$1 \wedge 1 = 0$

3) Disjunction / OR ($\vee$) :

$0 \vee 0 = 0$
$0 \vee 1 = 1$
$1 \vee 0 = 1$
$1 \vee 1 = 1$

4) Exclusive OR / XOR ($\oplus$) :

$0 \oplus 0 = 0$
$0 \oplus 1 = 1$
$1 \oplus 0 = 1$
$1 \oplus 1 = 0$

5) Equality ($\Leftrightarrow$)

$0 \Leftrightarrow 0 = 1$
$0 \Leftrightarrow 1 = 0$
$1 \Leftrightarrow 0 = 0$
$1 \Leftrightarrow 1 = 1$

WLOG = w/o loss of generality

$0 \Rightarrow 0 = 1$
$0 \Rightarrow 1 = 1$
$1 \Rightarrow 0 = 0$
$1 \Rightarrow 1 = 1$

c) implication ($\Rightarrow$)

a) Identities : $P \Rightarrow Q \equiv \neg P \vee Q$

$P \Leftrightarrow Q \equiv (P \Rightarrow Q) \wedge (Q \Rightarrow P)$

4. PROOFS

a) Techniques

1) construction.

2) contradiction.

- Thm : $\sqrt{2}$ is irrational,

ie $\nexists\, m, n \in \mathbb{Z}$ s.t. $\sqrt{2} = \frac{m}{n}$

- Pf/ By contradiction, Assume $\sqrt{2} = \frac{m}{n}$ for $m, n \in \mathbb{Z}$

① WLOG, either m or n is odd (o/w) divide top & bottom by 2)

② consider $\sqrt{2}n = m \Rightarrow 2n^2 = m^2$

↳ m^2 is even

$\Rightarrow m$ is even

③ ∴ $m = 2k$ for some $k \in \mathbb{Z}$

∴ $n^2 = m^2 \Rightarrow 2n^2 = (2k)^2$

$\Rightarrow 2n^2 = 4k^2$

$\Rightarrow n^2 = 2k^2$

↳ n^2 is even

$\Rightarrow n$ is even

∴ ②&③ conflict ① $\perp$

3) induction.

- used to prove properties about infinite sets.

e.g. Prove property P holds for $\mathbb{N} = \{1, 2, \dots\}$

STRUCTURE:

① Base case : prove $P(1)$ holds

② induction hypothesis (IH): Assume $P(i)$ hold for all $1 \le i \le k$

③ induction step: prove $P(k+1)$ holds

example
induction theorem

e.g. Thm $\forall n \in \mathbb{N}$, $1 + 3 + 5 + \dots + (2n-1) = n^2$.

Pf/ By Induction Base case: $(n=1)$: LHS 1, RHS $= (1)^2 = 1$ ✓

IH : Assume true for $1 \le n \le k$

induction step: prove true for $n = k+1$

LHS $1 + 3 + 5 + \dots + (2k-1) + (2(k+1)-1)$

$= \underbrace{1 + 3 + 5 + \dots + (2k-1)} + (2k+1)$

$= k^2 + (2k+1)$ IH

$= (k+1)^2$

Mathematical induction

Let $p(n)$ be a property defined on the integers and a be a fixed integer.

1) $P(a)$ is true.

2) For all integers $n \geq a$, if $p(n)$ is true then $P(n+1)$ is true.

Then $p(n)$ is true for all integers n with $n \geq a$.